

# Cycle-saturated graphs with minimum number of edges <sup>\*</sup>

**Zoltán Füredi** <sup>†</sup>

Department of Mathematics, University of Illinois at Urbana-Champaign,  
Urbana, IL 61801, USA    and

Rényi Institute of Mathematics of the Hungarian Academy of Sciences,  
Budapest, P. O. Box 127, Hungary-1364

e-mail: `z-furedi@illinois.edu`    and    `furedi@renyi.hu`

and

**Younjin Kim**

Department of Mathematics, University of Illinois at Urbana-Champaign,  
Urbana, IL 61801, USA.

e-mail: `ykim36@illinois.edu`

## Abstract

A graph  $G$  is called  $H$ -saturated if it does not contain any copy of  $H$ , but for any edge  $e$  in the complement of  $G$  the graph  $G + e$  contains some  $H$ . The minimum size of an  $n$ -vertex  $H$ -saturated graph is denoted by  $\text{sat}(n, H)$ . We prove

$$\text{sat}(n, C_k) = n + n/k + O((n/k^2) + k^2)$$

holds for all  $n \geq k \geq 3$ , where  $C_k$  is a cycle with length  $k$ . We have a similar result for semi-saturated graphs

$$\text{ssat}(n, C_k) = n + n/(2k) + O((n/k^2) + k).$$

We conjecture that our three constructions are optimal.

---

<sup>\*</sup>This copy was printed on January 13, 2013.    `cycle'sat'2011'0301.tex`,

Version as of March 01, 2011.

*Keywords:* graphs, cycles, extremal graphs, minimal saturated graphs.

*2010 Mathematics Subject Classification:* 05C38, 05C35.

<sup>†</sup>Research supported in part by the Hungarian National Science Foundation OTKA, and by the National Science Foundation under grant NFS DMS 09-01276.

## 1 A short history

A graph  $G$  is said to be  $H$ -saturated if

- it does not contain  $H$  as a subgraph, but
- the addition of any new edge (from  $E(\overline{G})$ ) creates a copy of  $H$ .

Let  $\text{sat}(n, H)$  denote the *minimum* size of an  $H$ -saturated graph on  $n$  vertices. Given  $H$ , it is difficult to determine  $\text{sat}(n, H)$  because this function is not necessarily monotone in  $n$ , or in  $H$ . Recent surveys are by J. Faudree, Gould, and Schmitt [11], and by Pikhurko [19] on the hypergraph case. It is known [17] that for every graph  $H$  there exists a constant  $c_H$  such that

$$\text{sat}(n, H) < c_H n$$

holds for all  $n$ . However, it is not known if the  $\lim_{n \rightarrow \infty} \text{sat}(n, H)/n$  exists; Pikhurko [19] has an example of a four graph set  $\mathcal{H}$  when  $\text{sat}(n, \mathcal{H})/n$  oscillates, it does not tend to a limit.

Since the classical theorem of Erdős, Hajnal, and Moon [9] (they determined  $\text{sat}(n, K_p)$  for all  $n$  and  $p$ ), and its generalization for hypergraphs by Bollobás [5], there have been many interesting hypergraph results (e.g., Kalai [16], Frankl [14], Alon [1], using Lovász' algebraic method) but here we only discuss the graph case.

Remarkable asymptotics were given by Alon, Erdős, Holzman, and Krivelevich [2, 10] (saturation and degrees). Bohman, Fonoberova, and Pikhurko [4] determined the sat-function asymptotically for a class of complete multipartite graphs. More recently, for multiple copies of  $K_p$  Faudree, Ferrara, Gould, and Jacobson [12] determined  $\text{sat}(tK_p, n)$  for  $n \geq n_0(p, t)$ .

## 2 Cycle-saturated graphs

What is the saturation number for the cycle,  $C_k$ ? This has been considered by various authors, however, in most cases it has remained unsolved. Here relatively tight bounds are given.

**Theorem 2.1.** *For all  $k \geq 7$  and  $n \geq 2k - 5$*

$$\left(1 + \frac{1}{k+2}\right)n - 1 < \text{sat}(n, C_k) < \left(1 + \frac{1}{k-4}\right)n + \binom{k-4}{2}. \quad (1)$$

The construction giving the upper bound is presented at the end of this section, the proof of the lower bound (which works for all  $n, k \geq 5$ ) is postponed to Section 10.

The case of  $\text{sat}(n, C_3) = n - 1$  is trivial; the cases  $k = 4$  and  $k = 5$  were established by Ollmann [18] in 1972 and by Ya-Chen [7] in 2009, resp.

$$\begin{aligned} \text{sat}(n, C_4) &= \left\lfloor \frac{3n-5}{2} \right\rfloor & \text{for } n \geq 5. \\ \text{sat}(n, C_5) &= \left\lceil \frac{10(n-1)}{7} \right\rceil & \text{for } n \geq 21. \end{aligned} \quad (2)$$

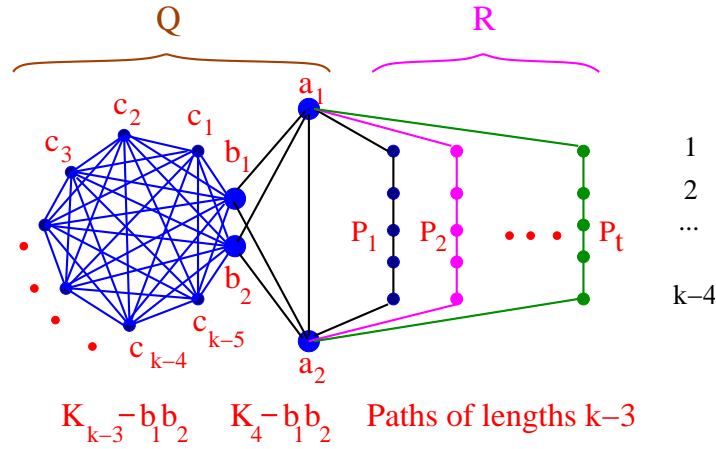
Actually, (2) was conjectured by Fisher, Fraughnaugh, Langley [13]. Later Ya-Chen [8] determined  $\text{sat}(n, C_5)$  for all  $n$ , as well as all extremal graphs.

The best previously known general lower bound came from Barefoot, Clark, Entringer, Porter, Székely, and Tuza [3], and the best upper bound (a clever, complicated construction resembling a bicycle wheel) came from Gould, Łuczak, and Schmitt [15]

$$\left(1 + \frac{1}{2k+8}\right)n \leq \text{sat}(n, C_k) \leq \left(1 + \frac{2}{k-\varepsilon(k)}\right)n + O(k^2) \quad (3)$$

where  $\varepsilon(k) = 2$  for  $k$  even  $\geq 10$ ,  $\varepsilon(k) = 3$  for  $k$  odd  $\geq 17$ . Although there is still a gap, Theorem 2.1 supersedes all earlier results for  $k \geq 6$  except the construction giving  $\text{sat}(n, C_6) \leq \frac{3}{2}n$  for  $n \geq 11$  from [15].

Our new construction for a  $k$ -cycle saturated graph for  $n = (k-1) + t(k-4)$  can be read from the picture below.



To be precise, define the graph  $H := H_{k,n}$  on  $n$  vertices, for arbitrary  $n > k \geq 7$  as follows. Write  $n$  in the form

$$n = (k-1) + r + t(k-4)$$

where  $t \geq 1$  is an integer and  $0 \leq r \leq k-5$ . The vertex set  $V(H)$  consists of the pairwise disjoint sets  $A$ ,  $B$ ,  $C$ ,  $D$ , and  $R_i$  for  $1 \leq i \leq t$ ,  $V(H) = A \cup B \cup C \cup D \cup R_1 \cup R_2 \cup \dots \cup R_t$  where  $|A| = |B| = 2$ ,  $|C| = k-5$ ,  $|D| = r$ , and  $|R_1| = |R_2| = \dots = |R_t| = k-4$  and  $A = \{a_1, a_2\}$ ,  $B = \{b_1, b_2\}$ ,  $C = \{c_1, c_2, \dots, c_{k-5}\}$ ,  $D = \{d_1, d_2, \dots, d_r\}$ ,  $R_\alpha = \{r_{\alpha,1}, r_{\alpha,2}, \dots, r_{\alpha,k-4}\}$ . We also denote  $A \cup B \cup C \cup D$  by  $Q$  and  $R_1 \cup \dots \cup R_t$  by  $R$ .

The edge set of  $H$  does not contain  $b_1 b_2$  and it consists of an almost complete graph  $K_{k-3}$  minus an edge on  $C \cup B$ , a  $K_4$  minus an edge on  $B \cup A$ ,  $r$  pending edges connecting  $c_i$  and  $d_i$ , and

$t$  paths  $P_\alpha$  of length  $k - 3$  with vertex sets  $A \cup R_\alpha$  with endpoints  $a_1$  and  $a_2$ . The number of edges

$$|E(G)| = \binom{k-3}{2} + 4 + r + t(k-3).$$

It is not difficult to check that, indeed,  $H$  is  $C_k$ -saturated (See details in Section 3). After which, a little calculation yields the upper bound in (1).

We strongly believe that this construction is essentially optimal.

**Conjecture 2.2.** *There exists a  $k_0$  such that  $\text{sat}(n, C_k) = \left(1 + \frac{1}{k-4}\right)n + O(k^2)$  holds for each  $k > k_0$ .*

### 3 The graph $H_{k,n}$ is $C_k$ -saturated, the proof of the upper bound for $\text{sat}(n, C_k)$

First we check that  $H := H_{k,n}$  is  $C_k$ -free. If a cycle with vertex set  $Y$  is entirely in  $Q$ , then it is contained in  $A \cup B \cup C$ , so  $|Y| \leq k - 1$ . If  $Y$  contains a vertex  $r_{\alpha,i}$  then  $A \cup R_\alpha \subset Y$ , the  $k - 3$  edges of the path  $P_\alpha$  are part of the cycle. However, it is impossible to join  $a_1$  and  $a_2$  by a path of length 3, so  $|Y| \neq k$ .

The key observation to know that  $H$  is  $C_k$ -saturated is that  $a_1$  and  $a_2$  are connected inside  $Q$  by a path  $T_\ell$  of any other lengths  $\ell$  except for 3

$$\exists \text{ path } T_\ell \subset Q : \ell \in \{1, 2, 4, 5, \dots, k-3, k-2\} \text{ with endpoints } a_1, a_2. \quad (4)$$

For example,  $T_1$  is  $a_1a_2$ ,  $T_2 = a_1b_1a_2$ ,  $T_4 = a_1b_1c_1b_2a_2$ , etc. Also the vertices  $a_i$  ( $i = 1, 2$ ) and  $q \in Q \setminus \{a_i\}$  are connected by a path  $U^i(m)$  of length  $m$  inside  $Q$  for  $\lceil (k+1)/2 \rceil \leq m \leq k-2$ .

$$\exists \text{ path } U^i(m) \subset Q : m \in \{\lceil (k+1)/2 \rceil, \dots, k-3, k-2\} \text{ with endpoints } a_i, q \in Q. \quad (5)$$

Note that this is true for any  $m \geq 4$  but we will apply (5) only for  $\lceil (k+1)/2 \rceil \geq 4$ .

Now add an edge  $e$  to  $H$  from its complement. We distinguish four disjoint cases.

Case 1. If  $e$  is contained in the induced cycle  $A \cup R_\alpha$  then we get a path connecting  $a_1$  and  $a_2$  in  $A \cup R_\alpha$  of length  $t$ , where  $t$  is at least two and at most  $k - 4$ . This path with  $T_{k-t}$  form a  $k$ -cycle.

Case 2. If the endpoints of  $e$  are  $r_{\alpha,i}$  and  $r_{\beta,j}$  with  $\alpha \neq \beta$  then we may suppose that  $1 \leq i \leq j \leq k - 4$ . The vertex  $r_{\alpha,i}$  splits the path  $P_\alpha$  into two parts,  $P_\alpha^1$  and  $P_\alpha^2$ , where  $P_\alpha^1$  starts at  $a_1$  and has length  $i$ , and  $P_\alpha^2$  ends at  $a_2$  and has length  $k - 3 - i$ . Consider the path  $\pi := P_\alpha^1 e P_\beta^2$ , its length is  $k - 2 - j + i$ . This length is between 3 and  $k - 2$  so we can apply (4) to add an appropriate  $T_{j-i+2}$  to complete  $\pi$  to a  $k$ -cycle unless  $j - i + 2 = 3$ . In the latter, the edge  $a_1a_2$  together with  $P_\beta^1$ ,  $e$ ,

and  $P_\alpha^2$  form a  $C_k$ .

Case 3. If the endpoints of  $e$  are  $r_{\alpha,i}$  and  $q \in B \cup C \cup D$ , then again by symmetry, we may suppose that  $i \leq (k-3)/2$ , so the length of  $P_\alpha^1$  is at most  $\lfloor (k-3)/2 \rfloor$ . Then, by (5) there is an  $U^1(m)$  so that  $P_\alpha^1$ ,  $e$  and  $U^1(m)$  form a  $k$ -cycle.

Case 4. Finally,  $e$  is contained in  $Q$ .

For  $e = a_1c_1$  we use  $P_1$  to get the  $k$ -cycle  $a_1c_1b_1a_2P_1$ ,

for  $e = a_1d_1$  we have the  $k$ -cycle  $d_1c_1c_2 \dots c_{k-5}b_2a_2b_1a_1$ ,

for  $e = b_1b_2$  we have to use  $P_1$ , i.e., here we need again that  $t \geq 1$ ,

for  $e = b_1d_1$  we have the  $k$ -cycle  $d_1c_1c_2 \dots c_{k-5}b_2a_2a_1b_1$ ,

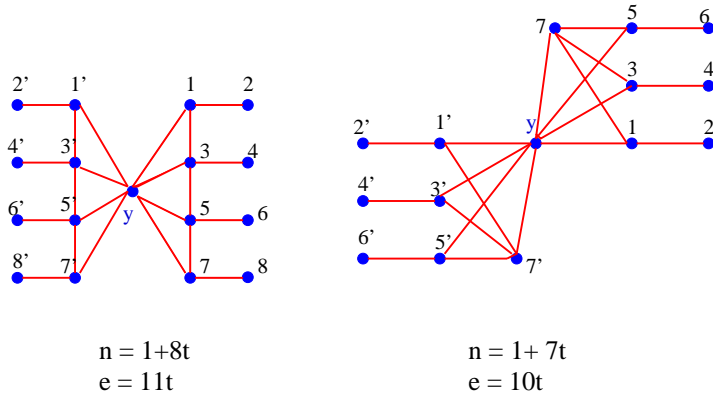
for  $e = c_1d_2$  we have the  $k$ -cycle  $c_1d_2c_2 \dots c_{k-5}b_2a_2a_1b_1$ , finally

for  $e = d_1d_2$  we have the  $k$ -cycle  $c_1d_1d_2c_2 \dots c_{k-5}b_2a_2b_1$ .  $\square$

## 4 Semisaturated graphs

A graph  $G$  is  **$H$ -semisaturated** (formerly called *strongly saturated*) if  $G + e$  contains more copies of  $H$  than  $G$  does for  $\forall e \in E(\overline{G})$ . Let  $\text{ssat}(n, H)$  be the minimum size of an  $H$ -semisaturated graph. Obviously,  $\text{ssat}(n, H) \leq \text{sat}(n, H)$ .

It is known that  $\text{ssat}(n, K_p) = \text{sat}(n, K_p)$  (it follows from Frankl/Alon/Kalai generalizations of Bollobás set pair theorem) and  $\text{ssat}(n, C_4) = \text{sat}(n, C_4)$  (Tuza [20]). Below we have a  $C_5$ -semisaturated graph on  $1 + 8t$  vertices and  $11t$  edges. Every vertex can be reached by a path of length 2 from  $y$ . Joining one, two or three triangles to the central vertex  $y$  one obtains  $C_5$



semisaturated graphs with  $8t + 3$ ,  $8t + 5$ , or  $8t + 7$  vertices and  $11t + 3$ ,  $11t + 6$ , or  $11t + 9$  edges, resp. Leaving out a pendant edge, we can extend these constructions for even values of  $n$

$$\text{ssat}(n, C_5) \leq \left\lceil \frac{11}{8}(n-1) \right\rceil \text{ for all } n \geq 5. \quad (6)$$

The picture on the right is the extremal  $C_5$ -saturated graph by (2).

**Conjecture 4.1.**  $\text{ssat}(n, C_5) = \frac{11}{8}n + O(1)$ . *Maybe equality holds in (6) for  $n > n_0$ .*

Since  $11/8 = 1.375 < 10/7 = 1.42\dots$  inequalities (2) and (6) imply that

$$\text{ssat}(n, C_5) < \text{sat}(n, C_5) \text{ for all } n \geq 21.$$

Our next Theorem shows that a similar statement holds for every cycle  $C_k$  with  $k > 12$  (and probably for  $k \in \{6, 7, \dots, 12\}$ , too).

**Theorem 4.2.** *For all  $n \geq k \geq 6$*

$$\left(1 + \frac{1}{2k-2}\right)n - 2 < \text{ssat}(n, C_k) < \left(1 + \frac{1}{2k-10}\right)n + k - 1. \quad (7)$$

The proof of the lower bound is postponed to Section 9. The construction yielding the upper bound is presented in the next two sections where we describe a way to improve the  $O(k)$  term as well as give better constructions for  $k = 6$ . We believe that our constructions are essentially optimal.

**Conjecture 4.3.** *There exists a  $k_0$  such that  $\text{ssat}(n, C_k) = \left(1 + \frac{1}{2k-10}\right)n + O(k)$  holds for each  $k > k_0$ .*

## 5 Constructions of sparse $C_k$ -semisaturated graphs

In this section we define an infinite class of  $C_k$ -semisaturated graphs,  $H_{k,n}^2$  (more precisely  $H_{k,n}^2(G)$ ).

Call a graph  $G$  *k-suitable* with special vertices  $a_1$  and  $a_2$  if

- (S1)  $G$  is  $C_k$ -semisaturated,
- (S2)  $\exists$  a path  $T_\ell$  in  $G$  with endpoints  $a_1$  and  $a_2$  and of length  $\ell$  for all  $1 \leq \ell \leq k-2$ , and
- (S3) for every  $q \in V(G) \setminus \{a_1, a_2\}$ , and integers  $m_1$  and  $m_2$  with  $m_1 + m_2 = k$  and  $2 \leq m_i \leq k-2$   
 $\exists$  an  $i \in \{1, 2\}$  and a path  $U(a_i, q, m_i)$  of length  $m_i$  and with endpoints  $a_i$  and  $q$ .

For example, it is easy to see, that a **wheel** with  $r$  spikes  $W_k^r$  is such a graph,  $k \geq r$ ,  $k \geq 4$ . It is defined by the  $(k+r)$ -element vertex set  $\{a_1, a_2, \dots, a_k, d_1, \dots, d_r\}$  and by  $2k-2+r$  edges joining  $a_1$  to all other  $a_i$ 's, forming a cycle  $a_2a_3 \dots a_k$  of length  $k-1$ , and joining each  $d_i$  to  $a_i$ .

Define the graph  $H_{k,n}^2(G)$  as follows, when  $n$  is in the form

$$n = |V(G)| + t(k-3)$$

where  $t \geq 0$  is an integer. The vertex set  $V(H)$  consists of the pairwise disjoint sets  $Q$  and  $R_i$  for  $1 \leq i \leq t$ ,  $V(H) = Q \cup R_1 \cup \dots \cup R_t$  where  $|Q| = |V(G)|$ ,  $|R_1| = |R_2| = \dots = |R_t| = k-3$  and

$A := \{a_1, a_2\} \subset Q$ . The edge set of  $H$  consists of a copy of  $G$  with vertex set  $Q$ , and  $t$  paths with endpoints  $a_1$  and  $a_2$  and vertex sets  $A \cup R_\alpha$ . The number of edges is

$$|E(H)| = |E(G)| + t(k - 2).$$

It is not difficult to check that, indeed,  $H$  is  $C_k$ -semisaturated, the details are similar (but much simpler) to those in Section 3, so we do not repeat that proof.

Finally, considering  $H_{k,n}^2(W_k^r)$  (where now  $4 \leq r \leq k$ ) we obtain that for all  $n \geq k + 4$

$$\text{ssat}(n, C_k) \leq n + \left\lfloor \frac{n-7}{k-3} \right\rfloor + k - 3. \quad (8)$$

**Corollary 5.1.**  $\text{ssat}(n, C_6) \leq \left\lceil \frac{4}{3}n \right\rceil$ .

## 6 Thinner constructions of sparse $C_k$ -semisaturated graphs

In this section we define another infinite class of  $C_k$ -semisaturated graphs,  $H_{k,n}^3$  (more precisely  $H_{k,n}^3(G)$ ) yielding the upper bound (7) in Theorem 4.2.

Call a graph  $G$   $\{k, k+2\}$ -suitable with special vertices  $a_1$  and  $a_2$  if (S1) and (S2) hold but (S3) is replaced by the following

(S3)<sup>+</sup> for every  $q \in V(G) \setminus \{a_1, a_2\}$ , and integers  $m_1, m_2$  either there exists a path  $U(a_1, q, m_1)$  (of length  $m_1$  and with endpoints  $a_1$  and  $q$ ) or a path  $U(a_2, q, m_2)$  in the following cases  
 $m_1 + m_2 = k$  and  $3 \leq m_i \leq k - 3$ ,  
 $m_1 + m_2 = k + 2$  and  $4 \leq m_i \leq k - 4$ .

It is easy to see, that the wheel  $W_k^r$  with  $r$  spikes is such a graph,  $k \geq r \geq 0$ ,  $k \geq 4$ .

Define the graph  $H_{k,n}^3(G)$  as follows, when  $n$  is in the form

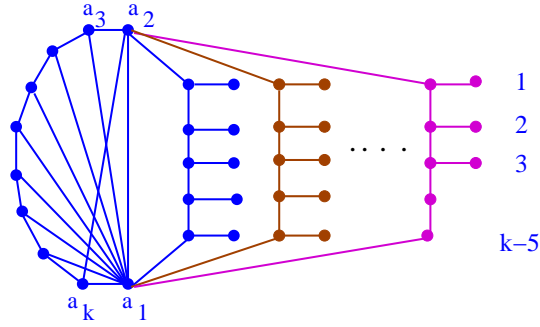
$$n = |V(G)| + t(2k - 10) - r \quad (9)$$

where  $t \geq 2$  is an integer and  $0 \leq r < 2k - 10$ . The vertex set  $V(H)$  consists of the pairwise disjoint sets  $Q$ ,  $R_i$  and  $D$  for  $1 \leq i \leq t$ ,  $V(H) = Q \cup R_1 \cup \dots \cup R_t \cup D$  where  $|Q| = |V(G)|$ ,  $|R_1| = |R_2| = \dots = |R_t| = k - 5$ ,  $|D| = t(k - 5) - r$  and  $A := \{a_1, a_2\} \subset Q$ . The edge set of  $H$  consists of a copy of  $G$  with vertex set  $Q$ , and  $t$  paths with endpoints  $a_1$  and  $a_2$  and vertex sets  $A \cup R_\alpha$  and finally  $|D|$  spikes, a matching with edges from  $\cup R_\alpha$  to  $D$ .

The number of edges is

$$|E(H)| = |E(G)| + t(2k - 9) - r. \quad (10)$$

It is not difficult to check that  $H$  is  $C_k$ -semisaturated, the details are similar (but simpler) to those in Section 3. As an example we present one case.



Add the edge  $qd$  to  $H$  where  $q \in V(G) \setminus \{a_1, a_2\}$  and  $d \in D$ . Let us denote the (unique) neighbor of  $d$  by  $x$ ,  $x \in R_\alpha$ . The distance of  $x$  to  $a_1$  is denoted by  $\ell$ . Then the length of the  $qdx \dots a_1$  path is  $\ell + 2 \geq 3$  and the length of the  $qdx \dots a_2$  path is  $(k - 4 - \ell) + 2 \geq 3$  and one can find a  $C_k$  through  $qd$  using property (S3)<sup>+</sup>.

Considering  $H_{k,n}^3(W_k)$  (with  $t \geq 2$ ) we obtain from (9) and (10) that for all  $n \geq 3k - 9$

$$\text{ssat}(n, C_k) \leq \left\lceil \left(1 + \frac{1}{2k - 10}\right) (n - k) \right\rceil + 2k - 2. \quad (11)$$

Using  $H^2(k, n)$ , it is easy to see that (11) holds for all  $n \geq k$ , leading to the upper bound in (7).

One can slightly improve (8) and (11) if there are special graphs thinner than the wheel  $W_k$ .

**Problem 6.1.** Determine  $s(k)$ , the minimum size of a  $k$ -vertex  $k$ -special graph (i.e., one satisfying (S1)–(S3)). Determine  $s'(k)$ , the minimum size of a  $k$ -vertex  $\{k, k + 2\}$ -special graph (i.e., one satisfying (S1), (S2) and (S3)<sup>+</sup>).

## 7 Degree one vertices in (semi)saturated graphs

Suppose that  $G$  is a  $C_k$ -semisaturated graph where  $k \geq 5$ ,  $|V(G)| = n \geq k$ . Obviously,  $G$  is connected. Let  $X$  be the set of vertices of degree one,  $X := \{v \in V(G) : \deg_G(v) = 1\}$ , its size is  $s$  and its elements are denoted as  $X = \{x_1, x_2, \dots, x_s\}$ . Denote the neighbor of  $x_i$  by  $y_i$ ,  $Y := \{y_1, \dots, y_s\}$  and let  $Z := V(G) \setminus (X \cup Y)$ . We also denote the neighborhood of any vertex  $v$  by  $N_G(v)$  or briefly by  $N(v)$ .

**Lemma 7.1.** (The neighbors of degree one vertices.)

- (i)  $y_i \neq y_j$  for  $1 \leq i \neq j \leq s$ , so  $|Y| = |X|$ .
- (ii)  $\deg(y) \geq 3$  for every  $y \in Y$ ,
- (iii) if  $\deg_G(x) = 1$ , then  $G - \{x\}$  is also a  $C_k$ -semisaturated graph.

*Proof.* If  $y_i = y_j$ , then the addition of  $x_i x_j$  to  $G$  does not create a new  $k$ -cycle. If  $\deg(y_i) = 2$  and  $N(y_i) = \{x_i, w\}$ , the addition of  $x_i w$  to  $G$  does not create a new  $k$ -cycle. Finally, (iii) is obvious.  $\square$



Split  $Y$  and  $Z$  according to the degrees of their vertices. Thus, divide  $V(G)$  into five parts  $\{X, Y_3, Y_{4+}, Z_2, Z_{3+}\}$ ,

$$Y_3 := \{v \in Y : \deg_G(v) = 3\} \text{ and } Y_{4+} := \{v \in Y : \deg_G(v) \geq 4\},$$

$$Z_2 := \{v \in Z : \deg_G(v) = 2\} \text{ and } Z_{3+} := \{v \in Z : \deg_G(v) \geq 3\}.$$

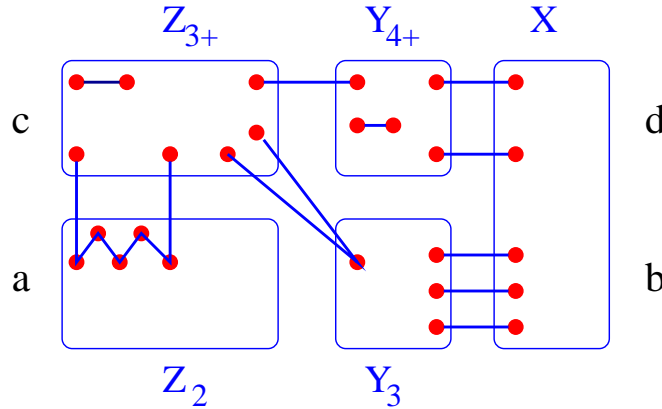
**Lemma 7.2.** (The structure of  $C_k$ -saturated graphs. See [3]).

Suppose that  $G$  is a  $C_k$ -saturated graph (and  $k \geq 5$ ). Then

(iv) if  $x_i y_i w$  is a path in  $G$  (with  $x_i \in X$ ,  $y_i \in Y$ ), then  $\deg(w) \geq 3$ . So there are no edges from  $Z_2$  to  $Y$  (or to  $X$ ).

(v) If  $y_i y_j$  is an edge of  $G$  (with  $y_i, y_j \in Y$ ), then  $\deg(y_i) \geq 4$ . So there are no edges in  $Y_3$ , no edges from  $Y_3$  to  $Y_{4+}$ . In other words, every  $y \in Y_3$  has one neighbor in  $X$  and two in  $Z_{3+}$ .

(vi) The induced graph  $G[Z_2]$  consists of paths of length at most  $k - 2$ . □



## 8 Semisaturated graphs without pendant edges

**Claim 8.1.** Suppose that  $G$  is a  $C_k$ -semisaturated graph on  $n$  vertices with minimum degree at least 2,  $k \geq 5$ . Then every vertex  $w$  is contained in some cycle of length at most  $k + 1$ .

*Proof.* Consider two arbitrary vertices  $z_1, z_2$  in the neighborhood  $N(w)$ . If  $z_1 z_2 \in E(G)$ , then  $w$  is contained in a triangle. If  $z_1 z_2 \notin E(G)$ , then  $G + z_1 z_2$  contains a new  $k$ -cycle; there is a path  $P$  of length  $(k - 1)$  in  $G$  with endpoints  $z_1$  and  $z_2$ . If  $P$  avoids  $w$ , then  $P$  together with  $z_1 w z_2$  form a  $k + 1$  cycle. If  $w$  splits  $P$  into two paths  $L_1, L_2$ , where  $L_i$  starts in  $z_i$  and ends in  $w$ , then either  $L_1 + z_1 w$ , or  $L_2 + z_2 w$ , or both form a proper cycle of length at most  $k - 1$ . □

Note that Claim 8.1 itself (and the connectedness of  $G$ ) immediately imply

$$e(G) \geq (n - 1) \frac{k + 2}{k + 1}.$$

We can do a bit better repeatedly using the semisaturatedness of  $G$ .

**Lemma 8.2.** *Suppose that  $G$  is a  $C_k$ -semisaturated graph on  $n$  vertices with minimum degree at least 2,  $k \geq 5$ . Then*

$$e(G) \geq \frac{k}{k-1}n - \frac{k+1}{k-1}.$$

*Proof.* We define an increasing sequence of subgraphs  $G_1, G_2, \dots, G_t = G$  with vertex sets  $V_1 \subseteq V_2 \subseteq \dots \subseteq V_t = V(G)$  such that  $G_i$  is a subgraph of  $G_{i+1}$  and

$$|E(G_{i+1}) \setminus E(G_i)| \geq \frac{k}{k-1} (|V_{i+1}| - |V_i|) \quad (12)$$

(for  $i = 1, 2, \dots, t-1$ ). This, together with

$$e(G_1) \geq \frac{k}{k-1} |V_1| - \frac{k+1}{k-1} \quad (13)$$

imply the claim.

$G_1$  is the shortest cycle in the graph  $G$ . Its length is at most  $k+1$  so (13) obviously holds.

If  $G_i$  is defined and one can find a path  $P$  of length at most  $k$  with endpoints in  $V_i$  but  $E(P) \setminus E(G_i) \neq \emptyset$ , then we can take  $E(G_{i+1}) = E(G_i) \cup E(P)$ . From now on, we suppose that such a *short returning* path does not exist. Our procedure stops if  $V(G_i) = V(G) =: V$ .

In the case of  $V \setminus V_i \neq \emptyset$ , the connectedness of  $G$  implies that there exists an  $xy$  edge with  $x \in V_i$  and  $y \in V \setminus V_i$ . Since  $|N(y)| \geq 2$  we have another edge  $yz \in E(G)$ ,  $z \neq x$ .

We have  $N(y) \cap V_i = \{x\}$ , otherwise one gets a path  $xyz$  of length smaller than  $k$  with endpoints in  $V_i$  but going out of  $G_i$ , contradicting our earlier assumption. Similarly, we obtain that  $N(y)$  contains no edge, otherwise we can define  $E(G_{i+1})$  as either  $E(G_i)$  plus the three edges of a triangle  $xy, yz, xz$  or we add four edges  $xy, yz_1, yz_2$ , and  $z_1z_2$  but only three vertices (namely  $y, z_1$ , and  $z_2$ ). The obtained  $G_{i+1}$  obviously satisfies (12) in both cases. Similarly, if there is a cycle  $C$  of length at most  $k-1$  containing  $y$ , then we can define  $E(G_{i+1})$  as  $E(G_i)$  plus  $E(C)$  and  $xy$ . From now on, we suppose that such a *short cycle through  $y$*  does not exist.

Fix a neighbor  $z$  of  $y$ ,  $z \neq x$ . Since  $zx \notin E(G)$ ,  $G$  contains a path  $P$  of length  $k-1$  with endvertices  $x$  and  $z$ . Since  $G$  does not contain a short returning path nor a short cycle through  $y$ , we obtain that  $P$  avoids  $y$  and  $V(P) \cap V_i = \{x\}$ .

If the cycle  $C := P + xy + yz$  of length  $k+1$  has any diagonal edge then  $G_{i+1}$  is obtained by adding  $C$  together with its diagonals. From now on, we suppose that  $C$  does not have any diagonals. More generally, if there is any *diagonal path*  $P$  of length  $\ell \leq k-1$  with edges disjoint from  $E(G_i) \cup E(C)$  but with endpoints in  $V_i \cup V(C)$  then we can define  $E(G_{i+1}) := E(G_i) \cup E(C) \cup E(P)$  and have added  $k + \ell - 1$  vertices and  $k + \ell + 1$  edges, obviously satisfying (12).

However, such a diagonal path exists. Let  $w \neq y$  be the other neighbor of  $x$  in  $C$ . Since  $wz \notin E(G)$ , there is a path  $P'$  of length  $k - 1$  with endpoints  $w$  and  $z$ . This  $P'$  must have edges outside  $E(G_i) \cup E(C)$  so it can be shortened to a diagonal path  $P$  of length at most  $k - 1$ . This completes the proof of the Lemma.  $\square$

## 9 A lower bound for the number of edges of semisaturated graphs

In this section we finish the proof of Theorem 4.2. Let  $G$  be a  $C_k$ -semisaturated graph on  $n$  vertices with minimum number of edges,  $k \geq 5$ . Let  $X$  be the set of degree one vertices,  $x := |X|$ . By Lemma 7.1  $|X| \leq n/2$ , and for  $G' := G \setminus X$  we have  $e(G') = e(G) - x$  and  $G'$  is a  $C_k$ -semisaturated graph on  $n - x$  vertices with minimum degree at least 2. Then Lemma 8.2 can be applied to  $e(G')$ . We obtain

$$\begin{aligned} \text{ssat}(n, C_k) = e(G) &\geq x + (n - x) \frac{k}{k-1} - \frac{k+1}{k-1} \\ &\geq \frac{n}{2} + \frac{n}{2} \frac{k}{k-1} - \frac{k+1}{k-1} = n \left( 1 + \frac{1}{2k-2} \right) - \frac{k+1}{k-1} \end{aligned}$$

$\square$

Since  $\text{sat}(n, C_k) \geq \text{ssat}(n, C_k)$ , this is already a better lower bound than the one in (3) from [3].

## 10 A lower bound for the number of edges of $C_k$ -saturated graphs

In this section we finish the proof of Theorem 2.1. Let  $G$  be a  $C_k$ -saturated graph on  $n$  vertices,  $k \geq 5$ . Let us consider the partition of  $V(G) = X \cup Y_3 \cup Y_{4+} \cup Z_2 \cup Z_{3+}$  defined in Section 7, where  $X$  is the set of degree one vertices,  $Y$  is their neighbors. By Lemma 7.1  $|X| = |Y|$ . To simplify notations we use  $a := |Z_2|$ ,  $b := |Y_3|$ ,  $c := |Z_{3+}|$ , and  $d := |Y_{4+}|$ . We have

$$n = a + 2b + c + 2d.$$

By definition of the parts we have the lower bound

$$2e(G) = \sum_{v \in V} \deg(v) \geq |X| + 2|Z_2| + 3|Y_3| + 3|Z_{3+}| + 4|Y_{4+}|.$$

This yields

$$2e \geq 2n + c + d. \tag{14}$$

Now we estimate the number of edges by considering four disjoint subsets of  $E(G)$ . The part  $X$  is adjacent to  $|X|$  edges, and according to Lemma 7.2,  $Z_2$  is adjacent to at least  $\frac{k}{k-1}|Z_2|$  edges,

$Y_3$  is adjacent to exactly  $3|Y_3|$  edges from which  $|Y_3|$  has already been counted at  $X$ , and finally  $Y_{4+}$  is adjacent to at least another  $\frac{3}{2}|Y_{4+}|$  edges. We obtain

$$e(G) \geq |X| + \frac{k}{k-1}|Z_2| + 2|Y_3| + \frac{3}{2}|Y_{4+}|.$$

Therefore we get

$$e \geq n + \frac{1}{k-1}a + b - c + \frac{1}{2}d. \quad (15)$$

By Lemma 7.1  $G \setminus X$  is also  $C_k$ -semisaturated. Apply Lemma 8.2 to estimate  $e(G \setminus X) = e - b - d$ , multiply by  $(k-1)$  and rearrange, we get

$$(k-1)e \geq kn - b - d - (k+1). \quad (16)$$

Adding up the above three inequalities (14), (15), and (16) we obtain

$$(k+2)e \geq (k+3)n + \frac{1}{k-1}a + \frac{1}{2}d - (k+1).$$

This implies the desired lower bound in (1).  $\square$

**Remark.** We can do slightly better if we multiply (14), (15), and (16) by  $k$ ,  $k-1$ , and  $k-3$ , resp., then adding up and simplifying we get

$$e(G) > \frac{k^2}{k^2 - k + 2} n - 1. \quad (17)$$

## References

- [1] Noga Alon, *An extremal problem for sets with applications to graph theory*, J. Combin. Theory Ser. A, **40** (1985), 82–89.
- [2] Noga Alon, Paul Erdős, Ron Holzman, Michael Krivelevich, *On  $k$ -saturated graphs with restrictions on the degrees*, J. Graph Theory, **23** (1996), 1–20.
- [3] C. A. Barefoot, L. H. Clark, R. C. Entringer, T. D. Porter, L. A. Székely, Zs. Tuza, *Cycle-saturated graphs of minimum size*, Discrete Mathematics **150** (1996), 31–48.
- [4] Tom Bohman, Maria Fonoferova, Oleg Pikhurko, *The saturation function of complete partite graphs*, J. Comb., **1** (2010), 149–170.
- [5] B. Bollobás, *On generalized graphs*, Acta Math. Acad. Sci. Hungar. **16** (1965), 447–452.
- [6] B. Bollobás, *Weakly  $k$ -saturated graphs*, In Beiträge zur Graphentheorie (Kolloquium, Manebach, 1967), pp. 25–31. Teubner, Leipzig, 1968.

- [7] Ya-Chen Chen, *Minimum  $C_5$ -saturated graphs*, J. Graph Theory **61** (2009), 111–126.
- [8] Ya-Chen Chen, *All minimum  $C_5$ -saturated graphs*, J. Graph Theory.
- [9] P. Erdős, A. Hajnal, J. W. Moon, *A problem in graph theory*, Amer. Math. Monthly **71** (1964), 1107–1110.
- [10] P. Erdős, Ron Holzman, *On maximal triangle-free graphs*, J. Graph Theory, **18** (1994), 585–594.
- [11] J. Faudree, R. Faudree, J. Schmitt, *A survey of minimum saturated graphs and hypergraphs*, manuscript, May 2010.
- [12] Ralph Faudree, Michael Ferrara, Ronald Gould, Michael Jacobson,  *$tK_p$ -saturated graphs of minimum size*, Discrete Math., **309** (2009), 5870–5876.
- [13] D. C. Fisher, K. Fraughnaugh, L. Langley, *On  $C_5$ -saturated graphs with minimum size*, Proceedings of the Twenty-sixth Southeastern International Conference on Combinatorics, Graph Theory and Computing (Boca Raton, FL, 1995), *Congr. Numer.* **112** (1995), 45–48.
- [14] Peter Frankl, *An extremal problem for two families of sets*, European J. Combin., **3** (1982), 125–127.
- [15] Ronald Gould, Tomasz Łuczak, John Schmitt, *Constructive upper bounds for cycle-saturated graphs of minimum size*, Electronic J. Combinatorics **13** (2006), Research Paper 29, 19 pp.
- [16] Gil Kalai, *Weakly saturated graphs are rigid*, In Convexity and graph theory (Jerusalem, 1981), volume 87 of North-Holland Math. Stud., pp. 189–190. North-Holland, Amsterdam, 1984.
- [17] L. Kászonyi, Zs. Tuza, *Saturated graphs with minimal numbers of edges*, J. Graph Theory **10** (1986), 203–210.
- [18] L. Taylor Ollmann,  *$K_{2,2}$  saturated graphs with a minimal number of edges*, In Proceedings of the Third Southeastern Conference on Combinatorics, Graph Theory, and Computing (Florida Atlantic Univ., Boca Raton, Fla., 1972), pages 367–392, Boca Raton, Fla., 1972. Florida Atlantic Univ.
- [19] O. Pikhurko, *Results and open problems on minimum saturated hypergraphs*, Ars Combin. **72** (2004), 111–127.
- [20] Zs. Tuza,  *$C_4$ -saturated graphs of minimum size*, Acta Univ. Carolin. Math. Phys., **30** (1989), 161–167.